Take home exam Geometry 7-4-2020

Always motivate your answers. You can freely use the results from the lecture notes. Please hand in your solutions in latex, scans of handwritten solutions will not be accepted. In addition the board of examiners has asked me to have you print, read, sign and scan the following declaration. Please send the signed declaration and your solutions to our email address `meetkunde20@gmail.com` before the deadline 21-4, 11pm. If some question is unclear or you believe there might be a typo, do not hesitate to contact us.
Good luck!

Declaration of the Board of Examiners

The Board of Examiners has allowed the conversion of your exam into a take-home exam. This conversion comes with additional provisions.

Here are the provisions that are relevant to you sitting the exam:
1. You are required to sign the attached pledge, swearing that your work has been completed autonomously and using only the tools and aids that the examiner has allowed you to use.
2. Attempts at cheating, fraud or plagiarism will be seen as attempts to take advantage of the Corona crisis and will be dealt with very harshly by the board of examiners.
3. The board of examiners grants your examiner the right to conduct a random sampling. If you are selected for this sample, you may be required to conduct a discussion (digitally, using audio and video) in which you are asked to explain and/or rephrase (some of) the answers you submitted for the take-home exam.

I, ...

(enter your name and student number here)

have completed this exam myself and without help from others unless expressly allowed by my lecturer. I have come up with these answers myself. I understand that my fellow students and my lecturers are all doing their best to do their work as well as possible under the unusual circumstances of the Corona pandemic, and that any attempt by myself or my fellow students to use these circumstances to get away with cheating would be undermining those efforts and the necessary trust that this moment calls for.

Signature:

1. In this exercise we consider an affine space $\mathcal{E}$ of dimension 2020.

   (a) Imagine three distinct parallel affine hyperplanes $\mathcal{H}, \mathcal{H}', \mathcal{H}'' \subset \mathcal{E}$ together with two affine lines $\mathcal{D}_1, \mathcal{D}_2$ whose directions are not contained in the direction of H of $\mathcal{H}$. Define the six points $P_i = \mathcal{H} \cap \mathcal{D}_i$, $P_i' = \mathcal{H}' \cap \mathcal{D}_i$ and $P_i'' = \mathcal{H}'' \cap \mathcal{D}_i$ for $i \in \{1, 2\}$. Prove that $\frac{\overrightarrow{P_1 P_1''}}{\overrightarrow{P_1 P_1'}} = \frac{\overrightarrow{P_2 P_2''}}{\overrightarrow{P_2 P_2'}}$.

   (b) Give an example of an affine map $\phi : \mathcal{E} \to \mathcal{E}$ that has no fixed points and is not a translation.

   (c) If $\mathcal{F}_1, \mathcal{F}_2$ are affine subspaces of $\mathcal{E}$ of equal dimension, is it true that there must exist an affine map $\phi : \mathcal{E} \to \mathcal{E}$ such that $\phi(\mathcal{F}_1) = \mathcal{F}_2$? Prove or give a counter example.

2. In the Euclidean affine plane $\mathcal{E}$ consider a circle F and two distinct points A, B on F.

   (a) For a point C on F consider the centroid G and orthocenter H of triangle $[A, B, C]$. If I is the midpoint of $[A, B]$ and O is the center of F then use the dilation $h_{G,-1/2}$ to prove $\overrightarrow{CH} = 2\overrightarrow{OI}$.

   (b) Define a function $f : F \to \mathcal{E}$ by setting $f(C)$ to be the orthocenter of triangle $[A, B, C]$. Prove that $f(F)$ equals the reflection of F in the line AB.

   (c) Apply Theorem 4.1 from the lecture notes to prove that if $a = d(B, C)$ is the length of the side opposite A of triangle $[A, B, C]$ inscribed in F (whose radius is R) and α is the measure of the geometric angle opposite to a then $\frac{a}{\sin \alpha} = 2R$.

3. Suppose E is a three-dimensional Euclidean vector space. By a half-twist we mean is a linear isometry of E that is a rotation whose oriented angle is the flat angle.

   (a) Prove that if s_J is the reflection in plane J through the origin then $-s_J$ is a half-twist. What is its axis?

   (b) Prove that for any two half-twists h, k there is an element $g \in O^+(E)$ such that $h = g \circ k \circ g^{-1}$.

   (c) Show that any $f \in O^+(E)$ may be written as a composition of finitely many half-twists.

4. We work in affine three-dimensional Euclidean space $\mathcal{E}$.

   (a) Show that the line segments connecting the mid-points of adjacent edges of a regular tetrahedron are precisely the edges of a regular octahedron.

(b) Explain how $\mathcal{E}$ is the union of countably many regular octahedra and tetrahedra of side length 1 in such a way that the intersection between any two polyhedra is empty, a single point, a single edge or a single face.

(c) If T is a regular tetrahedron with side length 1 and centroid O and $T' = h_{O,-1}(T)$ then describe the convex hull of $T \cap T'$ and also the convex hull of $T \cup T'$.

(d) Prove that there exists a sphere S passing through the midpoints of T and compute the spherical area of $S \cap T$.

5. Hyperbolic plane. In this exercise we identify $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$ in the standard way. Likewise a complex differentiable function $f : \mathbb{C} \to \mathbb{C}$ is identified with a differentiable function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ and multiplication by $f'(z)$ provides a linear map from $\mathbb{R}^2$ to itself that coincides with the derivative $\phi'(x, y)$. Next, the Euclidean inner product becomes $\langle v, w\rangle = \mathrm{Re}(v\bar{w})$.

   (a) For $a, b, c, d \in \mathbb{R}$ with $ad - bc = 1$ set $f(z) = \frac{az+b}{cz+d}$. Check that $f'(z) = (cz + d)^{-2}$ and $\mathrm{Im}f(z) = \mathrm{Im}(z)|cz + d|^{-2}$.

   (b) Identifying the set $\{z \in \mathbb{C} : \mathrm{Im}z > 0\}$ with the hyperbolic plane $\mathbb{H}$, show that the map f sends the hyperbolic plane to itself and that the hyperbolic metric g is written as $g(z)(v, w) = \frac{\mathrm{Re}(v\bar{w})}{(\mathrm{Im}z)^2}$.

   (c) Prove that the maps f (restricted to $\mathbb{H}$) are isometries in the sense that $g(f(z))(f'(z)v, f'(z)w) = g(z)(v, w)$.

   (d) Show that the straight line $Y = \{z \in \mathbb{H} : \mathrm{Re}z = 0\}$ is a geodesic.

   (e) Show that if $c, d \neq 0$ then $f(Y)$ is a Euclidean semi-circle with radius $\frac{1}{2cd}$ and center $\frac{1}{2}(f(0) + f(\infty))$ where $f(\infty) = \frac{a}{c}$.

   (f) Explain why $f(Y)$ is a geodesic.

   (g) Prove that all non-constant geodesics of $\mathbb{H}$ are Euclidean lines and semi-circles orthogonal to the real-axis.

   (h) For every $\pi > \epsilon > 0$ construct a triangle whose angle sum is ϵ. Angle sum means sum of measures of geometric angles with respect to the Riemannian metric.

   (i) Define $B = \{z \in \mathbb{H} : \mathrm{Re}(z) \in [0, 1], |z - \frac{1}{2}| \geq 1\}$. To emphasize the dependence of f on a, b, c, d we write f as $f_{a,b,c,d}$. Define $M = \{(a, b, c, d) \in \mathbb{Z} : ad - bc = 1\}$. Show that $\bigcup_{m\in M} f_m(B) = \mathbb{H}$ and that when $m \neq m'$ the intersection $f_m(B) \cap f_{m'}(B)$ is either a single geodesic or empty.

6. In the proof of Lemma 6.5 check explicitly that Δ satisfies the axioms of an LC-connection as claimed.

7. In this exercise we study Riemannian charts of the form $(P, \phi^* g_E)$ where $\phi : P \to \mathbb{R}^3$ is injective and C^2 with $P \subset \mathbb{R}^2$ some open set such that $\forall p \in P$: the derivative $\phi'(p)$ is injective.

(a) If $r : \mathbb{R}^3 \to \mathbb{R}^3$ is an affine Euclidean isometry and $\psi = r \circ \phi$ then prove that there is an isometry from $(P, \phi^* g_E)$ to $(P, \psi^* g_E)$.

(b) Suppose $\phi(x, y) = (x, y, f(x, y))$. Compute the scalar curvature of the Riemannian chart $(P, \phi^* g_E)$ at $(0, 0)$ in the case $f(x, y) = x^2 - y^2$.

(c) Prove for general C^2-functions $f : P \to \mathbb{R}$ such that $\partial_1 f(0, 0) = \partial_2 f(0, 0) = 0$ that the scalar curvature of the Riemannian chart $(P, \phi^* g_E)$ at point $(0, 0)$ is $2\mathrm{Hess}(f)(0, 0)$. Here the Hessian is $\mathrm{Hess}(f) = \det(\partial_i \partial_j f)$ is the determinant of the matrix of second partial derivatives.

(d) Define the (thick) Gauss map $G : P \times (-1, 1) \to \mathbb{R}^3$ sending a point in the thickened chart to the normal vector to the image of ϕ by

$$G(p, t) = (t + 1)\frac{\partial_1 \phi(p) \times \partial_2 \phi(p)}{|\partial_1 \phi(p) \times \partial_2 \phi(p)|}$$

Traditionally curvature is approached by studying how fast the normal vector turns. This is captured by the Gauss curvature at p which is $\det G'(p, 0)$. Under the same assumptions as in part c) prove that the scalar curvature at $(0, 0)$ equals twice the Gauss curvature at $(0, 0)$.